DD2552 Seminar 1: Functional languages, abstract syntax trees, variable binding, and inductive definitions

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1. Introduction
2. Motivating Examples
3. Abstract Syntax, Variable Binding, and Inductive Definitions

About me

- PhD KTH, 2014; postdoc/researcher 2015-2021
- lecturer, KTH, 2021-
- research areas:
  - program verification and proof engineering
  - distributed systems (e.g., blockchains)
- other teaching:
  - DD2443 Parallel and Distributed Computing
  - Prosam
- email available on Canvas

This course

- seminar course on purely functional programming
- spans from lambda calculus to efficient data structures
- material is useful in formal methods, but no fully formal proofs
- focus on principles and (some) practice

Course material

- **Practical Foundations of Programming Languages**
    - Robert Harper
    - 2nd Edition, Cambridge University Press, 2016
    - http://www.cs.cmu.edu/~rwh/pfpl/abbrev.pdf
- research papers
- CakeML language and compiler https://cakeml.org

What is a functional language?

- Common LISP, Scheme, Standard ML, OCaml, Haskell, …
- Erlang? Java and C++ due to lambda expressions?
- according to Robert Harper, a functional language should
    - give meaning to programs independently of its target (hardware or software) platform
    - support both **computation by evaluation** and computation by execution
    - support **persistent** and ephemeral data structures
    - have a parallel cost model
    - have a rich type structure, supporting modular program development
- https://existentialtype.wordpress.com/2011/03/16/what-is-a-functional-language/

Focus of this course

- program meaning in terms of mathematical functions and operational semantics
- computation by “pure” evaluation
- persistent data structures
- rich type structures, including modular structures

Why purely functional?

- referential transparency (substituting equals for equals works)
- heaps are difficult to reason about
- pure functions useful as specifications of other programs
- parallelizes easily
- useful for message passing concurrency (values don't change)
- performance can (still) be good to reasonable
- conjectured to be most feasible way to build large formally verified systems
- established mathematical theories and tool support (e.g., proof assistants such as Coq and HOL4)

Example System: CompCert C compiler

- compiler for a realistic subset of C (Misra C with extensions)
- core functionality defined as collection of pure functions
- language specification and machine-checked proofs of correctness in Coq proof assistant
- generated code generally performs better than gcc with O1 optimization
- useful in development of safety critical embedded systems
- ACM System Software Award 2022
- https://compcert.org

Example Language: Standard ML

- strongly typed functional language with full module system
- eager evaluation
- developed by Milner et al. in 1980s-1990s
- rigorously defined semantics
- several compilers, including Poly/ML with thread support
- impure, but easy to use pure fragment

```
fun factorial n =
 if n = 0 then 1 else n * factorial (n - 1)
```

Example Language: OCaml

- strongly typed functional language with full module system
- eager evaluation
- developed by Huet, Leroy et al. 1980s-present
- compiler defined semantics
- thread support in version 5
- impure, but easy to use pure fragment

```
let rec fact n =
 if n =/ Int 0 then Int 1 else n */ fact (n -/ Int 1)
```

Example Language: Haskell

- strongly typed functional language with typeclasses
- lazy evaluation
- developed by Peyton Jones et al. 1990s-present
- compiler defined semantics (GHC)
- thread support
- pure

```
factorial 0 = 1
factorial n = n * factorial (n - 1)
```

Example Language: CakeML

- strongly typed functional language with restricted module system
- eager evaluation
- developed by Myreen et al., 2010s-present
- fully formalized semantics, embedded in HOL4 proof assistant
- bootstrapped compiler with machine-checked correctness
- impure, but easy to use pure fragment
- no thread support

```
fun fib n =
 case n of
   0 => 0
 | 1 => 1
 | n => fib (n - 1) + fib (n - 2)
```

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Course material

- PFPL chapter 1 and chapter 2
- https://lawrencecpaulson.github.io/papers/Aczel-Inductive-Defs.pdf
- https://www.cs.cmu.edu/~rwh/pfpl/supplements/ulc.pdf

Object language vs. meta language

- in this course, we define syntax and semantics of some (small) languages
- the language being defined is usually called the **object language**
- the language we are using to define object languages is called the **meta language**
- our typical meta language is “mathematical English”, but could also be some foundational formalism like ZF set theory or constructive type theory
- the “ML” in Standard ML stands for meta language

Grammars and derivations

Consider the following grammar:

$$\begin{aligned} \text{Expr} \to \text{Num} \mid \text{Expr} \ + \ \text{Expr} \mid \text{Expr} \ - \ \text{Expr} \\ \mid \text{Expr} \ * \ \text{Expr} \mid \text{Expr} \, / \, \text{Expr} \mid (\text{Expr}) \end{aligned}$$

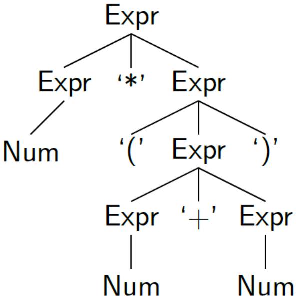
We can derive `"4*(3+5)"`:

$$\begin{aligned} \text{Expr} \to {} & \text{Expr} * \text{Expr} \to \text{Num} * \text{Expr} \to \text{Num} * (\text{Expr}) \to \\ & \text{Num} * (\text{Expr} + \text{Expr}) \to \text{Num} * (\text{Num} + \text{Expr}) \to \\ & \text{Num} * (\text{Num} + \text{Num}) \end{aligned}$$

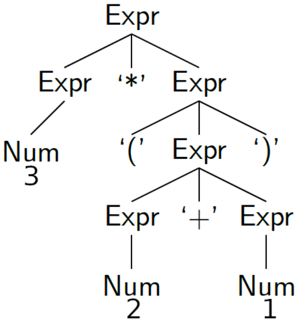
Abstract Syntax Trees

An abstract syntax tree (AST) can represent a set of derivations.

$$\begin{aligned}\underline{\text{Expr}} &\rightarrow \underline{\text{Expr}} * \text{Expr} \rightarrow \text{Num} * \underline{\text{Expr}} \rightarrow \text{Num} * (\underline{\text{Expr}}) \rightarrow \\ &\text{Num} * (\underline{\text{Expr}} + \text{Expr}) \rightarrow \text{Num} * (\text{Num} + \underline{\text{Expr}}) \rightarrow \\ &\text{Num} * (\text{Num} + \text{Num})\end{aligned}$$



Adding data to ASTs



ASTs and structural induction

Consider a more limited grammar:

$$\text{Expr} \to \text{Num} \mid \text{Expr} + \text{Expr}$$

We want to prove a property P holds for all ASTs in this grammar. It then suffices to:

- prove $P(n)$ for all numbers n
- assume $P(e)$ and $P(e')$ and prove $P(e + e')$

Why does this work? We cover all ways of forming strings according to grammar.

Variables and substitution

Consider a grammar with variables:

$$\mathrm{Expr} \to \mathrm{Num} \mid \mathrm{Expr} + \mathrm{Expr} \mid Var$$

If we have an expression e, we can *substitute* a variable x for a number n:

- $[b/x]x = b$ and $[b/x]y = y$ if $x \neq y$
- $[b/x]o(a_1, \dots, a_n) = o([b/x]a_1, \dots, [b/x]a_n)$

Variable binding is important

Let k be an unknown but fixed positive integer in the following definitions of sets:

- $L_1 = \{p^n q^n \mid n \leq k\}$
- $L_2 = \{p^n q^k \mid n > k\}$
- $L_3 = \{p^k q^n \mid k > n\}$
- $L_4 = \{p^n q^n r^n \mid n \leq k\}$
- $L_5 = \{p^n q^n r^n \mid n \geq k\}$

Variables binding in grammars

$$\text{Expr} \to \text{Num} \mid \text{Expr} + \text{Expr} \mid \text{Var} \mid \text{Let}\,\text{Var} = \text{Expr}\,\text{In}\,\text{Expr}$$

Now the second substitution rule won't work well anymore:

- $[b/x]x = b$ and $[b/x]y = y$ if $x \neq y$
- $[b/x]o(a_1, \dots, a_n) = o([b/x]a_1, \dots, [b/x]a_n)$

Solution: rename variables to avoid capture (see PFPL for details)

Inductive definitions (“judgments”)

$$\frac{\begin{array}{c} J_1 \\ \dots \\ J_n \end{array}}{J}$$

or

$$\frac{J_1 \dots J_n}{J}$$

- define relation or mathematical structure via rules
- relation name can occur in J_i (“recursive call”)
- rules can only be applied finite number of time

Inductive definition examples

$$\frac{}{n \Downarrow n} \quad \text{OS\_\_EVAL\_\_NUM}$$

$$\frac{\begin{array}{l} e_1 \Downarrow n_1 \\ e_2 \Downarrow n_2 \\ n_1 + n_2 = n \end{array}}{e_1 + e_2 \Downarrow n} \quad \text{OS\_\_EVAL\_\_PLUS}$$

Inductive derivations

- derive a judgment by reducing it to other judgments via rules
- all reductions must terminate using rules without (inductive) premises
- derivations are **trees** with the desired judgment (conclusion) as root

Ott

- tool for writing grammars and inductive rules
- exportable to LaTeX (also Coq, HOL4, Isabelle/HOL)
- https://github.com/ott-lang/ott

Ott grammar example

```
grammar
e :: e_ ::=
  | x :: :: var {{ com variable }}
  | n :: :: num {{ com number }}
  | e + e' :: :: plus {{ com plus }}
  | e * e' :: :: times {{ com times }}
  | let x := e in e' :: :: def (+ bind x in e' +)
    {{ com let }}
  | e [ e' / x ] :: M :: subst
    {{ com substitution }}
    {{ coq (subst_e [[e']] [[x]] [[e]]) }}
  | ( e ) :: S :: parentheses
    {{ coq ([[e]]) }}
```

Ott grammar using generated LaTeX

e	$::=$			
	$\|$	x		variable
	$\|$	n		number
	$\|$	$e + e'$		plus
	$\|$	$e * e'$		times
	$\|$	$\mathbf{let}\, x := e\, \mathbf{in}\, e'$	bind x in e'	let
	$\|$	$e[e'/x]$	M	substitution
	$\|$	(e)	S	

Ott rules example

```
defn
  e -> e' :: :: red :: red_
  {{ com reduction step }} by

  n1 + n2 = n
  ------------ :: plus
  n1 + n2 -> n

  e1 -> e'1
  ------------------- :: plus_l
  e1 + e2 -> e'1 + e2
```

Ott rules using generated LaTeX

$$\frac{n_1 + n_2 = n}{n_1 + n_2 \to n} \quad \text{OS\_\_RED\_\_PLUS}$$

$$\frac{e_1 \to e_1'}{e_1 + e_2 \to e_1' + e_2} \quad \text{OS\_\_RED\_\_PLUS\_\_L}$$

Semantics of expressions using rules

$$\frac{e \to e'}{n + e \to n + e'} \quad \text{OS\_\_RED\_\_PLUS\_\_R}$$

$$\frac{n_1 * n_2 = n}{n_1 * n_2 \to n} \quad \text{OS\_\_RED\_\_TIMES}$$

$$\frac{e_1 \to e_1'}{e_1 * e_2 \to e_1' * e_2} \quad \text{OS\_\_RED\_\_TIMES\_\_L}$$

$$\frac{e \to e'}{n * e \to n * e'} \quad \text{OS\_\_RED\_\_TIMES\_\_R}$$

$$\frac{e_1 \to e_1'}{\mathbf{let}\ x := e_1\ \mathbf{in}\ e_2 \to \mathbf{let}\ x := e_1'\ \mathbf{in}\ e_2} \quad \text{OS\_\_RED\_\_LET}$$

$$\frac{}{\mathbf{let}\ x := n\ \mathbf{in}\ e_2 \to e_2[n/x]} \quad \text{OS\_\_RED\_\_BIND}$$

Using the reduction relation

- we are given some expression AST e
- consider reflexive-transitive closure of $\rightarrow$ on expressions
- if $e \rightarrow^* n$, then n is the result of evaluating e
- to find and prove $e \rightarrow^* n$, we may have to do a lot of deriving

Alternative inductive relation

$$\frac{}{n \Downarrow n} \quad \text{OS\_\_EVAL\_\_NUM}$$

$$\frac{\begin{array}{c} e_1 \Downarrow n_1 \\ e_2 \Downarrow n_2 \\ n_1 + n_2 = n \end{array}}{e_1 + e_2 \Downarrow n} \quad \text{OS\_\_EVAL\_\_PLUS}$$

$$\frac{\begin{array}{c} e_1 \Downarrow n_1 \\ e_2 \Downarrow n_2 \\ n_1 * n_2 = n \end{array}}{e_1 * e_2 \Downarrow n} \quad \text{OS\_\_EVAL\_\_TIMES}$$

$$\frac{\begin{array}{c} e_1 \Downarrow n_1 \\ e_2[n_1/x] \Downarrow n_2 \end{array}}{\mathbf{let}\, x := e_1 \,\mathbf{in}\, e_2 \Downarrow n_2} \quad \text{OS\_\_EVAL\_\_LET}$$

How is this related to $\rightarrow$?