Chapter 1

Structural induction on lists

Definition of lists

$$List ::= \mathsf{empty} \mid \mathsf{cons}(Letter, List)$$

List length

Definition 1.1.

$$\mathbf{length}(\mathsf{empty}) \overset{\mathsf{def}}{=} 0$$
$$\mathbf{length}(\mathsf{cons}(a, u)) \overset{\mathsf{def}}{=} 1 + \mathbf{length}(u)$$

Exercise 1.1. Prove that

$$\forall a\ \forall b\ \mathbf{length}(\mathsf{cons}(a, \mathsf{cons}(b, \mathsf{empty}))) = 2.$$

Let a och b be *Letter* terms.

$$\begin{aligned}
&\mathbf{length}(\mathsf{cons}(a, \mathsf{cons}(b, \mathsf{empty}))) & \\
&= 1 + \mathbf{length}(\mathsf{cons}(b, \mathsf{empty})) & \{\text{Def. 1.1}\} \\
&= 1 + 1 + \mathbf{length}(\mathsf{empty}) & \{\text{Def. 1.1}\} \\
&= 1 + 1 + 0 & \{\text{Def. 1.1}\} \\
&= 2 & \{\text{Arithmetic}\}
\end{aligned}$$

Concatenation of lists

Definition 1.2.

$$\mathbf{conc}(\mathsf{empty}, v) \overset{\mathsf{def}}{=} v$$
$$\mathbf{conc}(\mathsf{cons}(a, u), v) \overset{\mathsf{def}}{=} \mathsf{cons}(a, \mathbf{conc}(u, v))$$

Exercise 1.2. Prove that

$$\forall a\ \forall b\ \mathbf{conc}(\mathsf{cons}(a, \mathsf{empty}), \mathsf{cons}(b, \mathsf{empty})) = \mathsf{cons}(a, \mathsf{cons}(b, \mathsf{empty}))$$

Let a and b be *Letter* terms.

$$
\begin{aligned}
&\mathbf{conc}(\mathsf{cons}(a, \mathsf{empty}), \mathsf{cons}(b, \mathsf{empty})) & \\
&= \mathsf{cons}(a, \mathbf{conc}(\mathsf{empty}, \mathsf{cons}(b, \mathsf{empty}))) & \{\text{Def. 1.2}\} \\
&= \mathsf{cons}(a, \mathsf{cons}(b, \mathsf{empty})) & \{\text{Def. 1.2}\}
\end{aligned}
$$

Exercise 1.3. Prove that

$$\forall u\ \mathbf{conc}(u, \mathsf{empty}) = u$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.

$$
\begin{aligned}
&\mathbf{conc}(\mathsf{empty}, \mathsf{empty}) & \\
&= \mathsf{empty} & \{\text{Def. 1.2}\}
\end{aligned}
$$

- Case $u = \mathsf{cons}(a, u')$ for a a *Letter* term and u' a *List* term.
  Assume (IH) that $\mathbf{conc}(u', \mathsf{empty}) = u'$.

$$
\begin{aligned}
&\mathbf{conc}(\mathsf{cons}(a, u'), \mathsf{empty}) & \\
&= \mathsf{cons}(a, \mathbf{conc}(u', \mathsf{empty})) & \{\text{Def. 1.2}\} \\
&= \mathsf{cons}(a, u') & \{\text{IH}\}
\end{aligned}
$$

Exercise 1.4. Prove that

$$\forall u\ \forall v\ \forall w\ \mathbf{conc}(u, \mathbf{conc}(v, w)) = \mathbf{conc}(\mathbf{conc}(u, v), w)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.
  Let v and w be *List* terms.

$$
\begin{aligned}
&\mathbf{conc}(\mathsf{empty}, \mathbf{conc}(v, w)) & \\
&= \mathbf{conc}(v, w) & \{\text{Def. 1.2}\} \\
&= \mathbf{conc}(\mathbf{conc}(\mathsf{empty}, v), w) & \{\text{Def. 1.2}\}
\end{aligned}
$$

- Case $u = \mathsf{cons}(a, u')$.
  Let v and w be *List* terms and assume (IH) that $\forall v\ \forall w\ \mathbf{conc}(u', \mathbf{conc}(v, w)) = \mathbf{conc}(\mathbf{conc}(u', v), w)$.

$$
\begin{aligned}
&\mathbf{conc}(\mathsf{cons}(a, u'), \mathbf{conc}(v, w)) & \\
&= \mathsf{cons}(a, \mathbf{conc}(u', \mathbf{conc}(v, w))) & \{\text{Def. 1.2}\} \\
&= \mathsf{cons}(a, \mathbf{conc}(\mathbf{conc}(u', v), w)) & \{\text{IH}\} \\
&= \mathbf{conc}(\mathsf{cons}(a, (\mathbf{conc}(u', v)), w) & \{\text{Def. 1.2}\} \\
&= \mathbf{conc}(\mathbf{conc}(\mathsf{cons}(a, u'), v), w) & \{\text{Def. 1.2}\}
\end{aligned}
$$

Reversal of lists

Initial reversal function

Definition 1.3.

$$\begin{aligned} \mathbf{reverse}(\mathsf{empty}) &\stackrel{\mathsf{def}}{=} \mathsf{empty} \\ \mathbf{reverse}(\mathsf{cons}(a, u)) &\stackrel{\mathsf{def}}{=} \mathbf{conc}(\mathbf{reverse}(u), \mathsf{cons}(a, \mathsf{empty})) \end{aligned}$$

Exercise 1.5. Prove that

$$\forall a\ \forall b\ \mathbf{reverse}(\mathsf{cons}(a, \mathsf{cons}(b, \mathsf{empty}))) = \mathsf{cons}(b, \mathsf{cons}(a, \mathsf{empty}))$$

Let a and b be *Letter* terms.

$$\begin{aligned} &\mathbf{reverse}(\mathsf{cons}(a, \mathsf{cons}(b, \mathsf{empty}))) \\ &= \mathbf{conc}(\mathbf{reverse}(\mathsf{cons}(b, \mathsf{empty})), \mathsf{cons}(a, \mathsf{empty})) && \{\text{Def. 1.3}\} \\ &= \mathbf{conc}(\mathbf{conc}(\mathbf{reverse}(\mathsf{empty}), \mathsf{cons}(b, \mathsf{empty})), \mathsf{cons}(a, \mathsf{empty})) && \{\text{Def. 1.3}\} \\ &= \mathbf{conc}(\mathbf{conc}(\mathsf{empty}, \mathsf{cons}(b, \mathsf{empty})), \mathsf{cons}(a, \mathsf{empty})) && \{\text{Def. 1.2}\} \\ &= \mathbf{conc}(\mathsf{cons}(b, \mathsf{empty}), \mathsf{cons}(a, \mathsf{empty})) && \{\text{Def. 1.2}\} \\ &= \mathsf{cons}(b, \mathbf{conc}(\mathsf{empty}, \mathsf{cons}(a, \mathsf{empty}))) && \{\text{Def. 1.2}\} \\ &= \mathsf{cons}(b, \mathsf{cons}(a, \mathsf{empty})) && \{\text{Def. 1.2}\} \end{aligned}$$

Exercise 1.6. Prove that

$$\forall u\ \forall v\ \mathbf{reverse}(\mathbf{conc}(u, v)) = \mathbf{conc}(\mathbf{reverse}(v), \mathbf{reverse}(u))$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.

  Let v be a *List* term.

$$\begin{aligned} &\mathbf{reverse}(\mathbf{conc}(\mathsf{empty}, v)) \\ &= \mathbf{reverse}(v) && \{\text{Def. 1.2}\} \\ &= \mathbf{conc}(\mathbf{reverse}(v), \mathsf{empty}) && \{\text{Ex. 1.3}\} \\ &= \mathbf{conc}(\mathbf{reverse}(v), \mathbf{reverse}(\mathsf{empty})) && \{\text{Def. 1.3}\} \end{aligned}$$

- Case $u = \mathsf{cons}(a, u')$.

Assume (IH) that $\forall v$ $\textbf{reverse}(\textbf{conc}(u', v)) = \textbf{conc}(\textbf{reverse}(v), \textbf{reverse}(u'))$, and let v be a *List* term.

$$\begin{aligned}
&\textbf{reverse}(\textbf{conc}(\textsf{cons}(a, u'), v)) \\
&= \textbf{reverse}(\textsf{cons}(a, \textbf{conc}(u', v))) && \{\text{Def. } 1.2\} \\
&= \textbf{conc}(\textbf{reverse}(\textbf{conc}(u', v)), \textsf{cons}(a, \textsf{empty})) && \{\text{Def. } 1.3\} \\
&= \textbf{conc}(\textbf{conc}(\textbf{reverse}(v), \textbf{reverse}(u')), \textsf{cons}(a, \textsf{empty})) && \{\text{IH}\} \\
&= \textbf{conc}(\textbf{reverse}(v), \textbf{conc}(\textbf{reverse}(u'), \textsf{cons}(a, \textsf{empty}))) && \{\text{Ex. } 1.4\} \\
&= \textbf{conc}(\textbf{reverse}(v), \textbf{reverse}(\textsf{cons}(a, u'))) && \{\text{Def. } 1.3\}
\end{aligned}$$

A better reversal function?

Definition 1.4.

$$\begin{aligned}
\textbf{rev}(\textsf{empty}, v) &\stackrel{\text{def}}{=} v \\
\textbf{rev}(\textsf{cons}(a, u), v) &\stackrel{\text{def}}{=} \textbf{rev}(u, \textsf{cons}(a, v)) \\
\textbf{reverse}'(u) &\stackrel{\text{def}}{=} \textbf{rev}(u, \textsf{empty})
\end{aligned}$$

Exercise 1.7. Prove that

$$\forall a\ \forall b\ \textbf{reverse}'(\textsf{cons}(a, \textsf{cons}(b, \textsf{empty}))) = \textsf{cons}(b, \textsf{cons}(a, \textsf{empty}))$$

Let a and b be *Letter* terms.

$$\begin{aligned}
&\textbf{reverse}'(\textsf{cons}(a, \textsf{cons}(b, \textsf{empty}))) \\
&= \textbf{rev}(\textsf{cons}(a, \textsf{cons}(b, \textsf{empty})), \textsf{empty}) && \{\text{Def. } 1.4\} \\
&= \textbf{rev}(\textsf{cons}(b, \textsf{empty}), \textsf{cons}(a, \textsf{empty})) && \{\text{Def. } 1.4\} \\
&= \textbf{rev}(\textsf{empty}, \textsf{cons}(b, \textsf{cons}(a, \textsf{empty}))) && \{\text{Def. } 1.4\} \\
&= \textsf{cons}(b, \textsf{cons}(a, \textsf{empty})) && \{\text{Def. } 1.4\}
\end{aligned}$$

Exercise 1.8. Prove that

$$\forall u\ \forall v\ \textbf{conc}(\textbf{reverse}'(u), v) = \textbf{rev}(u, v)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \textsf{empty}$.

  Let v be a *List* term.

$$\begin{aligned}
&\textbf{conc}(\textbf{reverse}'(\textsf{empty}), v) \\
&= \textbf{conc}(\textbf{rev}(\textsf{empty}, \textsf{empty}), v) && \{\text{Def. } 1.4\} \\
&= \textbf{conc}(\textsf{empty}, v) && \{\text{Def. } 1.4\} \\
&= v && \{\text{Def. } 1.2\} \\
&= \textbf{rev}(\textsf{empty}, v) && \{\text{Def. } 1.4\}
\end{aligned}$$

- Case $u = \mathsf{cons}(a, u')$.

  Assume (IH) that $\forall v\ \mathbf{conc}(\mathbf{reverse}'(u'), v) = \mathbf{rev}(u', v)$, and let v be a *List* term.

$$
\begin{aligned}
&\mathbf{conc}(\mathbf{reverse}'(\mathsf{cons}(a, u')), v) \\
&= \mathbf{conc}(\mathbf{rev}(u', \mathsf{cons}(a, \mathsf{empty})), v) && \{\text{Def. 1.4}\} \\
&= \mathbf{conc}(\mathbf{conc}(\mathbf{rev}(u', \mathsf{empty}), \mathsf{cons}(a, \mathsf{empty})), v) && \{\text{IH, Def. 1.4}\} \\
&= \mathbf{conc}(\mathbf{rev}(u', \mathsf{empty}), \mathbf{conc}(\mathsf{cons}(a, \mathsf{empty}), v)) && \{\text{Ex. 1.4}\} \\
&= \mathbf{conc}(\mathbf{reverse}'(u'), \mathsf{cons}(a, v)) && \{\text{Def. 1.4, Def. 1.2}\} \\
&= \mathbf{rev}(\mathsf{cons}(a, u'), v) && \{\text{IH, Def. 1.4}\}
\end{aligned}
$$

Exercise 1.9. Prove that

$$\forall u\ \mathbf{reverse}(u) = \mathbf{reverse}'(u)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.

$$
\begin{aligned}
&\mathbf{reverse}(\mathsf{empty}) \\
&= \mathsf{empty} && \{\text{Def. 1.3}\} \\
&= \mathbf{rev}(\mathsf{empty}, \mathsf{empty}) && \{\text{Def. 1.4}\} \\
&= \mathbf{reverse}'(\mathsf{empty}) && \{\text{Def. 1.4}\}
\end{aligned}
$$

- Case $u = \mathsf{cons}(a, u')$.

  Assume (IH) that $\mathbf{reverse}(u') = \mathbf{reverse}'(u')$.

$$
\begin{aligned}
&\mathbf{reverse}(\mathsf{cons}(a, u')) \\
&= \mathbf{conc}(\mathbf{reverse}(u'), \mathsf{cons}(a, \mathsf{empty})) && \{\text{Def. 1.3}\} \\
&= \mathbf{conc}(\mathbf{reverse}'(u'), \mathsf{cons}(a, \mathsf{empty})) && \{\text{IH}\} \\
&= \mathbf{reverse}'(\mathsf{cons}(a, u')) && \{\text{Ex. 1.8}\}
\end{aligned}
$$

Efficiency analysis

Measuring efficency

To enable measuring the efficiency of a function, we assume redefine the function to return both a cost and the original result.

Definition 1.5.

$$
\begin{aligned}
\mathbf{cost}(\langle s, d\rangle) &\stackrel{\text{def}}{=} s \\
\mathbf{result}(\langle s, d\rangle) &\stackrel{\text{def}}{=} d
\end{aligned}
$$

Measurable versions of functions

Consider a measurable version of **conc**.

Definition 1.6.

$$\begin{aligned}\textbf{mconc}(\mathsf{empty}, v) &\overset{\text{def}}{=} \langle 0, v\rangle \\ \textbf{mconc}(\mathsf{cons}(a, u'), v) &\overset{\text{def}}{=} \textbf{let } r = \textbf{mconc}(u', v) \textbf{ in} \\ &\quad\ \langle 1 + \textbf{cost}(r), \mathsf{cons}(a, \textbf{result}(r))\rangle\end{aligned}$$

Exercise 1.10. Prove that

$$\forall u\ \forall v\ \textbf{result}(\textbf{mconc}(u, v)) = \textbf{conc}(u, v)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.

  Let v be a *List* term.

$$\begin{aligned}&\textbf{result}(\textbf{mconc}(\mathsf{empty}, v)) \\ &= \textbf{result}(\langle 0, v\rangle) && \{\text{Def. 1.6}\} \\ &= v && \{\text{Def. 1.5}\} \\ &= \textbf{conc}(\mathsf{empty}, v) && \{\text{Def. 1.2}\}\end{aligned}$$

- Case $u = \mathsf{cons}(a, u')$.

  Assume (IH) that $\forall v\ \textbf{result}(\textbf{mconc}(u', v)) = \textbf{conc}(u', v)$, and let v be a *List* term.

$$\begin{aligned}&\textbf{result}(\textbf{mconc}(\mathsf{cons}(a, u'))) \\ &= \mathsf{cons}(a, \textbf{result}(\textbf{mconc}(u', v))) && \{\text{Def. 1.6, Def. 1.5}\} \\ &= \mathsf{cons}(a, \textbf{conc}(u', v)) && \{\text{IH}\} \\ &= \textbf{conc}(\mathsf{cons}(a, u'), v) && \{\text{Def. 1.2}\}\end{aligned}$$

Exercise 1.11. (A) Prove that

$$\forall u\ \forall v\ \textbf{length}(\textbf{result}(\textbf{mconc}(u, v))) = \textbf{length}(u) + \textbf{length}(v)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.

  Let v be a *List* term.

$$\begin{aligned}&\textbf{length}(\textbf{result}(\textbf{mconc}(\mathsf{empty}, v))) \\ &= \textbf{length}(\textbf{result}(\langle 0, v\rangle)) && \{\text{Def. 1.6}\} \\ &= \textbf{length}(v) && \{\text{Def. 1.5}\} \\ &= 0 + \textbf{length}(v) && \{\text{Arithmetic}\} \\ &= \textbf{length}(\mathsf{empty}) + \textbf{length}(v) && \{\text{Def. 1.1}\}\end{aligned}$$

- Case $u = \mathsf{cons}(a, u')$ for a a *Letter* term and u' a *List* term.

  Assume (IH) that $\forall v\ \mathbf{length}(\mathbf{result}(\mathbf{mconc}(u', v))) = \mathbf{length}(u') + \mathbf{length}(v)$, and let v be a *List* term.

$$
\begin{aligned}
&\mathbf{length}(\mathbf{result}(\mathbf{mconc}(\mathsf{cons}(a, u'), v))) \\
&= \mathbf{length}(\mathsf{cons}(a, \mathbf{result}(\mathbf{mconc}(u', v)))) && \{\text{Def. 1.6, Def. 1.5}\} \\
&= 1 + \mathbf{length}(\mathbf{result}(\mathbf{mconc}(u', v))) && \{\text{Def. 1.1}\} \\
&= 1 + \mathbf{length}(u') + \mathbf{length}(v) && \{\text{IH}\} \\
&= \mathbf{length}(\mathsf{cons}(a, u')) + \mathsf{length}(v) && \{\text{Def. 1.1}\}
\end{aligned}
$$

Exercise 1.12. Prove that

$$\forall u\ \forall v\ \mathbf{cost}(\mathbf{mconc}(u, v)) = \mathbf{length}(u)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.

  Let v be a *List* term.

$$
\begin{aligned}
&\mathbf{cost}(\mathbf{mconc}(\mathsf{empty}, v)) \\
&= \mathbf{cost}(\langle 0, v\rangle) && \{\text{Def. 1.6}\} \\
&= 0 && \{\text{Def. 1.5}\} \\
&= \mathbf{length}(\mathsf{empty}) && \{\text{Def. 1.1}\}
\end{aligned}
$$

- Case $u = \mathsf{cons}(a, u')$ for a a *Letter* term and u' a *List* term.

  Assume (IH) that $\forall v\ \mathbf{cost}(\mathbf{mconc}(u', v)) = \mathbf{length}(u')$, and let v be a *List* term.

$$
\begin{aligned}
&\mathbf{cost}(\mathbf{mconc}(\mathsf{cons}(a, u'), v)) \\
&= 1 + \mathbf{cost}(\mathbf{mconc}(u', v)) && \{\text{Def. 1.6, Def. 1.5}\} \\
&= 1 + \mathbf{length}(u') && \{\text{IH}\} \\
&= \mathbf{length}(\mathsf{cons}(a, u')) && \{\text{Def. 1.1}\}
\end{aligned}
$$

Definition 1.7.

$$
\begin{aligned}
\mathbf{mreverse}(\mathsf{empty}) &\stackrel{\mathsf{def}}{=} \langle 0, \mathsf{empty}\rangle \\
\mathbf{mreverse}(\mathsf{cons}(a, u')) &\stackrel{\mathsf{def}}{=} \mathbf{let}\ rr = \mathbf{mreverse}(u')\ \mathbf{in} \\
&\quad\ \mathbf{let}\ rc = \mathbf{mconc}(\mathbf{result}(rr), \mathsf{cons}(a, \mathsf{empty}))\ \mathbf{in} \\
&\quad\ \langle 1 + \mathbf{cost}(rc) + \mathbf{cost}(rr), \mathbf{result}(rc)\rangle
\end{aligned}
$$

Exercise 1.13. Prove that

$$\forall u\ \mathbf{result}(\mathbf{mreverse}(u)) = \mathbf{reverse}(u)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.

$$
\begin{aligned}
&\mathbf{result}(\mathbf{mreverse}(\mathsf{empty})) \\
&= \mathbf{result}(\langle 0, \mathsf{empty}\rangle) && \{\text{Def. 1.7}\} \\
&= \mathsf{empty} && \{\text{Def. 1.5}\} \\
&= \mathbf{reverse}(\mathsf{empty}) && \{\text{Def. 1.3}\}
\end{aligned}
$$

- Case $u = \mathsf{cons}(a, u')$.

  Assume (IH) that $\mathbf{result}(\mathbf{mreverse}(u')) = \mathbf{reverse}(u')$.

$$
\begin{aligned}
&\mathbf{result}(\mathbf{mreverse}(\mathsf{cons}(a, u'))) \\
&= \mathbf{result}(\mathbf{mconc}(\mathbf{result}(\mathbf{mreverse}(u')), \mathsf{cons}(a, \mathsf{empty}))) && \{\text{Def. 1.6}\} \\
&= \mathbf{conc}(\mathbf{result}(\mathbf{mreverse}(u')), \mathsf{cons}(a, \mathsf{empty})) && \{\text{Ex. 1.10}\} \\
&= \mathbf{conc}(\mathbf{reverse}(u'), \mathsf{cons}(a, \mathsf{empty})) && \{\text{IH}\} \\
&= \mathbf{reverse}(\mathsf{cons}(a, u')) && \{\text{Def. 1.3}\}
\end{aligned}
$$

Exercise 1.14. Prove that

$$\forall u \; \mathbf{length}(\mathbf{result}(\mathbf{mreverse}(u))) = \mathbf{length}(u)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.

$$
\begin{aligned}
&\mathbf{length}(\mathbf{result}(\mathbf{mreverse}(\mathsf{empty}))) \\
&= \mathbf{length}(\mathbf{result}(\langle 0, \mathsf{empty}\rangle) && \{\text{Def. 1.7}\} \\
&= \mathbf{length}(\mathsf{empty}) && \{\text{Def. 1.5}\}
\end{aligned}
$$

- Case $u = \mathsf{cons}(a, u')$.

  Assume (IH) that $\mathbf{length}(\mathbf{result}(\mathbf{mreverse}(u'))) = \mathbf{length}(u')$.

$$
\begin{aligned}
&\mathbf{length}(\mathbf{result}(\mathbf{mreverse}(\mathsf{cons}(a, u')))) \\
&= \mathbf{length}(\mathbf{result}(\mathbf{mconc}(\mathbf{result}(\mathbf{mreverse}(u')), \mathsf{cons}(a, \mathsf{empty})))) && \{\text{Def. 1.7}\} \\
&= \mathbf{length}(\mathbf{result}(\mathbf{mreverse}(u'))) + \mathbf{length}(\mathsf{cons}(a, \mathsf{empty})) && \{\text{Ex. 1.11}\} \\
&= 1 + \mathbf{length}(\mathbf{result}(\mathbf{mreverse}(u'))) && \{\text{Aritmetik}\} \\
&= 1 + \mathbf{length}(u') && \{\text{IH}\} \\
&= \mathbf{length}(\mathsf{cons}(a, u')) && \{\text{Def. 1.1}\}
\end{aligned}
$$

Exercise 1.15. Prove that

$$\forall u \; 2 \times \mathbf{cost}(\mathbf{mreverse}(u)) = \mathbf{length}(u) \times \mathbf{length}(u) + \mathbf{length}(u)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.

$$
\begin{aligned}
& 2 \times \mathbf{cost}(\mathbf{mreverse}(\mathsf{empty})) \\
& = 2 \times \mathbf{cost}(\langle 0, \mathsf{empty} \rangle) && \{\text{Def. 1.7}\} \\
& = 0 && \{\text{Def. 1.5}\} \\
& = 0 \times 0 + 0 && \{\text{Aritmetik}\} \\
& = \mathbf{length}(\mathsf{empty}) \times \mathbf{length}(\mathsf{empty}) + \mathbf{length}(\mathsf{empty}) && \{\text{Def. 1.1}\}
\end{aligned}
$$

- Case $u = \mathsf{cons}(a, u')$.
  Assume (IH) that $2 \times \mathbf{cost}(\mathbf{mreverse}(u')) = \mathbf{length}(u') \times \mathbf{length}(u') + \mathbf{length}(u')$.

$$
\begin{aligned}
& 2 \times \mathbf{cost}(\mathbf{mreverse}(\mathsf{cons}(a, u'))) \\
& = 2 \times (1 + \mathbf{length}(\mathbf{result}(\mathbf{mreverse}(u'))) + \mathbf{cost}(\mathbf{mreverse}(u'))) && \{\text{Def. 1.7, Ex. 1.12}\} \\
& = 2 \times (1 + \mathbf{length}(u') \ + \mathbf{cost}(\mathbf{mreverse}(u'))) && \{\text{Ex. 1.14}\} \\
& = 2 \times (1 + \mathbf{length}(u')) + 2 \times \mathbf{cost}(\mathbf{mreverse}(u')) && \{\text{Aritmetik}\} \\
& = 2 \times (1 + \mathbf{length}(u')) + \mathbf{length}(u') \times \mathbf{length}(u') + \mathbf{length}(u') && \{\text{IH}\} \\
& = (1 + \mathbf{length}(u')) \times (1 + \mathbf{length}(u')) + (1 + \mathbf{length}(u')) && \{\text{Aritmetik}\} \\
& = \mathbf{length}(\mathsf{cons}(a, u')) \times \mathbf{length}(\mathsf{cons}(a, u')) + \mathbf{length}(\mathsf{cons}(a, u')) && \{\text{Def. 1.7}\}
\end{aligned}
$$

Definition 1.8.

$$
\begin{aligned}
\mathbf{mrev}(\mathsf{empty}, v) &\overset{\mathsf{def}}{=} \langle 0, v \rangle \\
\mathbf{mrev}(\mathsf{cons}(a, u')) &\overset{\mathsf{def}}{=} \mathbf{let}\ r = \mathbf{mrev}(u', \mathsf{cons}(a, v))\ \mathbf{in} \\
&\quad\ \ \langle 1 + \mathbf{cost}(r), \mathbf{result}(r) \rangle \\
\mathbf{mreverse}'(u) &\overset{\mathsf{def}}{=} \mathbf{mrev}(u, \mathsf{empty})
\end{aligned}
$$

Exercise 1.16. Prove that

$$\forall u\ \forall v\ \mathbf{result}(\mathbf{mrev}(u, v)) = \mathbf{rev}(u, v)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.
  Let v be a *List* term.

$$
\begin{aligned}
& \mathbf{result}(\mathbf{mrev}(\mathsf{empty}, v)) \\
& = \mathbf{result}(\langle 0, v \rangle) && \{\text{Def. 1.8}\} \\
& = v && \{\text{Def. 1.5}\} \\
& = \mathbf{rev}(\mathsf{empty}, v) && \{\text{Def. 1.4}\}
\end{aligned}
$$

- Case $u = \mathsf{cons}(a, u')$.

Assume (IH) that $\forall v\ \mathbf{result}(\mathbf{mrev}(u', v)) = \mathbf{rev}(u', v)$, and let v be a *List* term.

$$
\begin{aligned}
&\mathbf{result}(\mathbf{mrev}(\mathsf{cons}(a, u'), v)) && \\
&= \mathbf{result}(\mathbf{mrev}(u', \mathsf{cons}(a, v))) && \{\text{Def. 1.8}\} \\
&= \mathbf{rev}(u', \mathsf{cons}(a, v)) && \{\text{IH}\}
\end{aligned}
$$

Exercise 1.17. Prove that

$$\forall u\ \mathbf{result}(\mathbf{mreverse}'(u)) = \mathbf{reverse}'(u)$$

Let u be a *List* term.

$$
\begin{aligned}
&\mathbf{result}(\mathbf{mreverse}'(u)) && \\
&= \mathbf{rev}(u, \mathsf{empty}) && \{\text{Ex. 1.16}\} \\
&= \mathbf{reverse}'(u) && \{\text{Def 1.4}\}
\end{aligned}
$$

Exercise 1.18. (A) Prove that

$$\forall u\ \forall v\ \mathbf{cost}(\mathbf{mrev}(u, v)) = \mathbf{length}(u)$$

Let u be a *List* term. We perform induction on the structure of u.

- Case $u = \mathsf{empty}$.
  Let v be a *List* term.

$$
\begin{aligned}
&\mathbf{cost}(\mathbf{mrev}(\mathsf{empty}, v)) && \\
&= \mathbf{cost}(\langle 0, v\rangle) && \{\text{Def. 1.8}\} \\
&= 0 && \{\text{Def. 1.5}\} \\
&= \mathbf{length}(\mathsf{empty}) && \{\text{Def. 1.1}\}
\end{aligned}
$$

- Case $u = \mathsf{cons}(a, u')$.
  Assume (IH) that $\forall v\ \mathbf{cost}(\mathbf{mrev}(u', v)) = \mathbf{length}(u')$, and let v be a *List* term.

$$
\begin{aligned}
&\mathbf{cost}(\mathbf{mrev}(\mathsf{cons}(a, u'), v) && \\
&= 1 + \mathbf{cost}(\mathbf{mrev}(u', \mathsf{cons}(a, v))) && \{\text{Def. 1.8, Def. 1.5}\} \\
&= 1 + \mathbf{length}(u') && \{\text{IH}\} \\
&= \mathbf{length}(\mathsf{cons}(a, u')) && \{\text{Def. 1.1}\}
\end{aligned}
$$

Exercise 1.19. Prove that

$$\forall u\ \mathbf{cost}(\mathbf{mreverse}'(u)) = \mathbf{length}(u)$$

Let u be a *List* term.

$$\begin{aligned} & \mathbf{cost}(\mathbf{mreverse}'(u)) & \\ & = \mathbf{cost}(\mathbf{mrev}(u, \mathsf{empty})) & \{\text{Def } 1.8\} \\ & = \mathbf{length}(u) & \{\text{Ex. } 1.18\} \end{aligned}$$