This document is based on material from the "Interactive Theorem Proving Course" by Thomas Tuerk (`https://www.thomas-tuerk.de`):
`https://github.com/thtuerk/ITP-course`

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Interactive Theorem Proving and Program Verification
Lecture 3

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Based on slides by Thomas Tuerk

Part VII

Backward Proofs

Motivation I

- let's prove `!A B. A /\ B <=> B /\ A`

```
(* Show |- A /\ B ==> B /\ A *)
val thm1a = ASSUME ``A /\ B``;
val thm1b = CONJ (CONJUNCT2 thm1a) (CONJUNCT1 thm1a);
val thm1  = DISCH ``A /\ B`` thm1b

(* Show |- B /\ A ==> A /\ B *)
val thm2a = ASSUME ``B /\ A``;
val thm2b = CONJ (CONJUNCT2 thm2a) (CONJUNCT1 thm2a);
val thm2  = DISCH ``B /\ A`` thm2b

(* Combine to get |- A /\ B <=> B /\ A *)
val thm3  = IMP_ANTISYM_RULE thm1 thm2

(* Add quantifiers *)
val thm4  = GENL [``A:bool``, ``B:bool``] thm3
```

- this is how you write down a proof you already know
- for finding a proof it is however often useful to think **backwards**

Motivation II - thinking backwards

- we want to prove
  - `!A B. A /\ B <=> B /\ A`
- all-quantifiers can easily be added later, so let's get rid of them
  - `A /\ B <=> B /\ A`
- now we have an equivalence, let's show 2 implications
  - `A /\ B ==> B /\ A`
  - `B /\ A ==> A /\ B`
- we have an implication, so we can use the precondition as an assumption
  - using `A /\ B` show `B /\ A`
  - `A /\ B ==> B /\ A`

Motivation III - thinking backwards

- we have a conjunction as assumption, let's split it
  - using `A` and `B` show `B /\ A`
  - `A /\ B ==> B /\ A`
- we have to show a conjunction, so let's show both parts
  - using `A` and `B` show `B`
  - using `A` and `B` show `A`
  - `A /\ B ==> B /\ A`
- the first two proof obligations are trivial
  - `A /\ B ==> B /\ A`
- ...
- we are done

Motivation IV

- common practise
  - think backwards to find proof
  - write found proof down in forward style
- often switch between backward and forward style within a proof
  Example: induction proof
  - backward step: induct on ...
  - forward steps: prove base case and induction case
- whether to use forward or backward proofs depend on
  - support by the interactive theorem prover you use
    - HOL4 and close family: emphasis on backward proof
    - Isabelle/HOL: emphasis on forward proof
    - Coq : emphasis on backward proof
  - your way of thinking
  - the theorem you try to prove

HOL4 Implementation of Backward Proofs

- in HOL4
  - proof tactics / backward proofs used for most user-level proofs
  - forward proofs used usually for writing automation
- backward proofs are implemented by **tactics** in HOL4
  - decomposition into subgoals implemented in SML
  - SML data structures used to keep track of all open subgoals
  - forward proof used to construct theorems
- to understand backward proofs in HOL4 we need to look at
  - `goal` — SML datatype for proof obligations
  - `goalStack` — library for keeping track of goals
  - `tactic` — SML type for functions performing backward proofs

Goals

- goals represent proof obligations, i. e. theorems we need/want to prove
- the SML type `goal` is an abbreviation for `term list * term`
- the goal `([asm_1, ..., asm_n], c)` records that we need/want to prove the theorem `{asm_1, ..., asm_n} |- c`

Example Goals

Goal	**Theorem**
``([``A``, ``B``], ``A /\ B``)``	`{A, B} |- A /\ B`
``([``B``, ``A``], ``A /\ B``)``	`{A, B} |- A /\ B`
``([``B /\ A``], ``A /\ B``)``	`{B /\ A} |- A /\ B`
``([], ``(B /\ A) ==> (A /\ B)``)``	`|- (B /\ A) ==> (A /\ B)`

Tactics

- the SML type `tactic` is an abbreviation for the type `goal -> goal list * validation`
- `validation` is an abbreviation for `thm list -> thm`
- given a goal, a tactic
  - decides into which subgoals to decompose the goal
  - returns this list of subgoals
  - returns a validation that
    - given a list of theorems for the computed subgoals
    - produces a theorem for the original goal
- special case: empty list of subgoals
  - the validation (given `[]`) needs to produce a theorem for the goal
- notice: a tactic might be invalid

Tactic Example — CONJ_TAC

$$\frac{\Gamma \vdash p \qquad \Delta \vdash q}{\Gamma \cup \Delta \vdash p \ \wedge \ q}\ \text{CONJ}$$

$$\frac{\begin{array}{c}\texttt{t} \equiv \texttt{conj1 /\textbackslash\ conj2} \\ \texttt{asl} \vdash \texttt{conj1} \qquad \texttt{asl} \vdash \texttt{conj2}\end{array}}{\texttt{asl} \vdash \texttt{t}}$$

```
val CONJ_TAC: tactic = fn (asl, t) =>
   let
      val (conj1, conj2) = dest_conj t
   in
      ([(asl, conj1), (asl, conj2)],
       fn [th1, th2] => CONJ th1 th2 | _ => raise Match)
   end
   handle HOL_ERR _ => raise ERR "CONJ_TAC" ""
```

Tactic Example — EQ_TAC

$$\frac{\Gamma \vdash p \Longrightarrow q \qquad \Delta \vdash q \Longrightarrow p}{\Gamma \cup \Delta \vdash p = q}\ \mathrm{IMP\_ANTISYM\_RULE}$$

$$\frac{\begin{array}{c}\texttt{t} \equiv \texttt{lhs = rhs} \\ \texttt{asl} \vdash \texttt{lhs ==> rhs} \\ \texttt{asl} \vdash \texttt{rhs ==> lhs}\end{array}}{\texttt{asl} \vdash \texttt{t}}$$

```
val EQ_TAC: tactic = fn (asl, t) =>
   let
      val (lhs, rhs) = dest_eq t
   in
      ([(asl, mk_imp (lhs, rhs)), (asl, mk_imp (rhs, lhs))],
       fn [th1, th2] => IMP_ANTISYM_RULE th1 th2
        | _          => raise Match)
   end
   handle HOL_ERR _ => raise ERR "EQ_TAC" ""
```

proofManagerLib / goalStack

- the `proofManagerLib` keeps track of open goals
- it uses `goalStack` internally
- important commands
  - **g** — set up new goal
  - **e** — expand a tactic
  - **p** — print the current status
  - **top_thm** — get the proved thm at the end

Tactic Proof Example I

Previous Goalstack

-

User Action

```
g `!A B. A /\ B <=> B /\ A`;
```

New Goalstack

```
Initial goal:

!A B. A /\ B <=> B /\ A

: proof
```

Tactic Proof Example II

Previous Goalstack

```
Initial goal:

!A B. A /\ B <=> B /\ A

: proof
```

User Action

```
e GEN_TAC;
e GEN_TAC;
```

New Goalstack

```
A /\ B <=> B /\ A

: proof
```

Tactic Proof Example III

Previous Goalstack

```
A /\ B <=> B /\ A


: proof
```

User Action

```
e EQ_TAC;
```

New Goalstack

```
B /\ A ==> A /\ B


A /\ B ==> B /\ A


: proof
```

Tactic Proof Example IV

Previous Goalstack

```
B /\ A ==> A /\ B


A /\ B ==> B /\ A : proof
```

User Action

```
e STRIP_TAC;
```

New Goalstack

```
  0.  A
  1.  B
------------------------------------
B /\ A
```

Tactic Proof Example V

Previous Goalstack

```
   0.  A
   1.  B
------------------------------------
B /\ A
```

User Action

```
e CONJ_TAC;
```

New Goalstack

```
   0.  A
   1.  B
------------------------------------
A

  0.  A
  1.  B
------------------------------------
B
```

Tactic Proof Example VI

Previous Goalstack

```
   0.  A
   1.  B
------------------------------------
A

  0.  A
  1.  B
------------------------------------
B
```

User Action

```
e (ACCEPT_TAC (ASSUME ``B:bool``));
e (ACCEPT_TAC (ASSUME ``A:bool``));
```

New Goalstack

```
B /\ A ==> A /\ B

: proof
```

Tactic Proof Example VII

Previous Goalstack

```
B /\ A ==> A /\ B

: proof
```

User Action

```
e STRIP_TAC;
e (ASM_REWRITE_TAC[]);
```

New Goalstack

```
Initial goal proved.
|- !A B. A /\ B <=> B /\ A:
   proof
```

Tactic Proof Example VIII

Previous Goalstack

```
Initial goal proved.
|- !A B. A /\ B <=> B /\ A:
   proof
```

User Action

```
val thm = top_thm();
```

Result

```
val thm =
   |- !A B. A /\ B <=> B /\ A:
   thm
```

Writing proof scripts in a file

The **prove** function

```
prove : term * tactic -> thm
```

- Takes a boolean term and attempts to prove it with the supplied tactic
- Fails with an exception if the tactic cannot solve the goal

Tactic Proof Example IX

Combined Tactic

```
val thm = prove (``!A B. A /\ B <=> B /\ A``,
  GEN_TAC >> GEN_TAC >>
  EQ_TAC >| [
    STRIP_TAC >>
    STRIP_TAC >| [
      ACCEPT_TAC (ASSUME ``B:bool``),
      ACCEPT_TAC (ASSUME ``A:bool``)
    ],

    STRIP_TAC >>
    ASM_REWRITE_TAC[]
  ]);
```

Result

```
val thm =
   |- !A B. A /\ B <=> B /\ A:
   thm
```

Tactic Proof Example X

Cleaned-up Tactic

```
val thm = prove (``!A B. A /\ B <=> B /\ A``,
  REPEAT GEN_TAC >>
  EQ_TAC >> (
    REPEAT STRIP_TAC >>
    ASM_REWRITE_TAC []
  ));
```

Result

```
val thm =
   |- !A B. A /\ B <=> B /\ A:
   thm
```

Summary Backward Proofs

- in HOL4 most user-level proofs are tactic-based
  - automation often written in forward style
  - low-level, basic proofs written in forward style
  - nearly everything else is written in backward (tactic) style
- there are **many** different tactics
- in the lecture only the most basic ones will be discussed
- you need to learn about tactics on your own
  - good starting point: `Quick` manual
  - learning finer points takes a lot of time
  - exercises require you to read up on tactics
- often there are many ways to prove a statement, which tactics to use depends on
  - personal way of thinking
  - personal style and preferences
  - maintainability, clarity, elegance, robustness
  - ...

Part VIII

Basic Tactics

Syntax of Tactics in HOL4

- originally tactics were written all in capital letters with underscores
  Example: `ALL_TAC`
- since 2010 more and more tactics have overloaded lower-case syntax
  Example: `all_tac`
- sometimes, the lower-case version is shortened
  Example: `REPEAT`, `rpt`
- sometimes, there is special syntax
  Example: `THEN`, `\\`, `>>`
- which one to use is mostly a matter of personal taste
  - all-capital names are hard to read and type
  - however, not for all tactics there are lower-case versions
  - mixed lower- and upper-case tactics are even harder to read
  - often shortened lower-case name is not *speaking*

In the lecture we will use mostly the old-style names.

Some Basic Tactics

`GEN_TAC`	remove outermost all-quantifier
`DISCH_TAC`	move antecedent of goal into assumptions
`CONJ_TAC`	splits conjunctive goal
`STRIP_TAC`	splits on outermost connective (combination of `GEN_TAC`, `CONJ_TAC`, `DISCH_TAC`, ...)
`DISJ1_TAC`	selects left disjunct
`DISJ2_TAC`	selects right disjunct
`EQ_TAC`	reduce Boolean equality to implications
`ASSUME_TAC` thm	add theorem to list of assumptions
`EXISTS_TAC` term	provide witness for existential goal

Tacticals

- tacticals are SML functions that combine tactics to form new tactics
- common workflow
  - develop large tactic interactively
  - using `goalStack` and editor support to execute tactics one by one
  - combine tactics manually with tacticals to create larger tactics
  - finally end up with one large tactic that solves your goal
  - use `prove` or `store_thm` instead of `goalStack`
- make sure to clearly mark proof structure by e. g.
  - use indentation
  - use parentheses
  - use appropriate connectives
  - ...
- goalStack commands like `e` or `g` should not appear in your final proof

Some Basic Tacticals

tac1 `>>` tac2	`THEN`, `\\`	applies tactics in sequence
tac `>|` tacL	`THENL`	applies list of tactics to subgoals
tac1 `>-` tac2	`THEN1`	applies tac2 to the first subgoal of tac1
`REPEAT` tac	`rpt`	repeats tac until it fails
`NTAC` n tac		apply tac n times
`REVERSE` tac	`reverse`	reverses the order of subgoals
tac1 `ORELSE` tac2		applies tac1 only if tac2 fails
`TRY` tac		do nothing if tac fails
`ALL_TAC`	`all_tac`	do nothing
`NO_TAC`		fail

Basic Rewrite Tactics

- (equational) rewriting is at the core of HOL4's automation
- we will discuss it in detail later
- details complex, but basic usage is straightforward
  - given a theorem `rewr_thm` of form `|- P x = Q x` and a term `t`
  - rewriting `t` with `rewr_thm` means
  - replacing each occurrence of a term `P c` for some `c` with `Q c` in `t`
- warning: rewriting may loop
  Example: rewriting with theorem `|- X <=> (X /\ T)`

`REWRITE_TAC` thms	rewrite goal using equations found in given list of theorems
`ASM_REWRITE_TAC` thms	in addition use assumptions
`ONCE_REWRITE_TAC` thms	rewrite once in goal using equations
`ONCE_ASM_REWRITE_TAC` thms	rewrite once using assumptions

Case-Split and Induction Tactics

`Induct_on` ‘term‘	induct on `term`
`Induct`	induct on all-quantifier
`Cases_on` ‘term‘	case-split on `term`
`Cases`	case-split on all-quantifier
`MATCH_MP_TAC` thm	apply rule
`IRULE_TAC` thm	generalised apply rule

Assumption Tactics

`POP_ASSUM` thm-tac	use and remove first assumption common usage `POP_ASSUM MP_TAC`
`PAT_ASSUM` term thm-tac also `PAT_X_ASSUM` term thm-tac	use (and remove) first assumption matching pattern
`WEAKEN_TAC` term-pred	removes first assumption satisfying predicate

Decision Procedure Tactics

- decision procedures try to solve the current goal completely
- they either succeed or fail
- no partial progress
- decision procedures vital for automation

`TAUT_TAC`	propositional logic tautology checker
`DECIDE_TAC`	linear arithmetic for `num`
`METIS_TAC` thms	first order prover
`numLib.ARITH_TAC`	Presburger arithmetic
`intLib.ARITH_TAC`	uses Omega test

Subgoal Tactics

- it is vital to structure your proofs well
  - improved maintainability
  - improved readability
  - improved reusability
  - saves time in medium-run
- therefore, use many small lemmata
- also, use many explicit subgoals

‘term-frag‘ `by` tac	show term with tac and add it to assumptions
‘term-frag‘ `suffices_by` tac	show it suffices to prove term

Term Fragments / Term Quotations

- notice that `by` and `suffices_by` take **term fragments**
- term fragments are also called **term quotations**
- they represent (partially) unparsed terms
- parsing takes place during execution of tactic in context of goal
- this helps to avoid type annotations
- however, this means syntax errors show late as well
- the library **Q** defines many tactics using term fragments

Importance of Exercises and Homeworks

- here many tactics are presented in a short amount of time
- there are many, many more tactics out there
- few people can learn a programming language just by reading manuals
- similarly, few people can learn HOL4 just by reading and listening
- you should write your own proofs and play around with these tactics

Tactical Proof - Example I - Slide 1

- we want to prove `!l. LENGTH (APPEND l l) = 2 * LENGTH l`
- first step: set up goal on `goalStack`
- at same time start writing proof script

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``,
```

Actions

- run `g ``!l. LENGTH (APPEND l l) = 2 * LENGTH l```
- this is done by hol-mode
- move cursor inside term and press `M-h g`
  (menu-entry `HOL - Goalstack - New goal`)

Tactical Proof - Example I - Slide 2

Current Goal

```
!l. LENGTH (l ++ l) = 2 * LENGTH l
```

- the outermost connective is an all-quantifier
- let's get rid of it via `GEN_TAC`

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (l ++ l) = 2 * LENGTH l``,
GEN_TAC
```

Actions

- run `e GEN_TAC`
- this is done by hol-mode
- mark line with `GEN_TAC` and press `M-h e`
  (menu-entry `HOL - Goalstack - Apply tactic`)

Tactical Proof - Example I - Slide 3

Current Goal

`LENGTH (l ++ l) = 2 * LENGTH l`

- `LENGTH` of `APPEND` can be simplified
- let's search an appropriate lemma with `DB.match`

Actions

- run `DB.print_match [] ``LENGTH (_ ++ _)```
- this is done via hol-mode
- press `M-h m` and enter term pattern
  (menu-entry `HOL - Misc - DB match`)
- this finds the theorem `listTheory.LENGTH_APPEND`
  `|- !l1 l2.  LENGTH (l1 ++ l2) = LENGTH l1 + LENGTH l2`

Tactical Proof - Example I - Slide 4

Current Goal

```
LENGTH (l ++ l) = 2 * LENGTH l
```

- let's rewrite with found theorem `listTheory.LENGTH_APPEND`

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``,
GEN_TAC >>
REWRITE_TAC[listTheory.LENGTH_APPEND]
```

Actions

- connect the new tactic with tactical `>>` (`THEN`)
- use hol-mode to expand the new tactic

Tactical Proof - Example I - Slide 5

Current Goal

```
LENGTH l + LENGTH l = 2 * LENGTH l
```

- let's search a theorem for simplifying `2 * LENGTH l`
- prepare for extending the previous rewrite tactic

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``,
GEN_TAC >>
REWRITE_TAC[listTheory.LENGTH_APPEND]
```

Actions

- `DB.match` finds theorem `arithmeticTheory.TIMES2`
- press `M-h b` and undo last tactic expansion (menu-entry `HOL - Goalstack - Back up`)

Tactical Proof - Example I - Slide 6

Current Goal

```
LENGTH (l ++ l) = 2 * LENGTH l
```

- extend the previous rewrite tactic
- finish proof

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``,
GEN_TAC >>
REWRITE_TAC[listTheory.LENGTH_APPEND, arithmeticTheory.TIMES2]);
```

Actions

- add `TIMES2` to the list of theorems used by rewrite tactic
- use hol-mode to expand the extended rewrite tactic
- goal is solved, so let's add closing parenthesis and semicolon

Tactical Proof - Example I - Slide 7

- we have a finished tactic proving our goal
- notice that `GEN_TAC` is not needed
- let's polish the proof script

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``,
GEN_TAC >>
REWRITE_TAC[listTheory.LENGTH_APPEND, arithmeticTheory.TIMES2]);
```

Polished Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``,
REWRITE_TAC[listTheory.LENGTH_APPEND, arithmeticTheory.TIMES2]);
```

Tactical Proof - Example II - Slide 1

- let's prove something slightly more complicated
- drop old goal by pressing `M-h d`
  (menu-entry `HOL - Goalstack - Drop goal`)
- set up goal on `goalStack` (`M-h g`)
- at same time start writing proof script

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (``!x1 x2 x3 l1 l2 l3.
  (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\
  ((x1 <= x2) /\ (x2 <= x3) /\ x3 <= SUC x1) ==>
  ~(ALL_DISTINCT (l1 ++ l2 ++ l3))``,
```

Tactical Proof - Example II - Slide 2

Current Goal

```
!x1 x2 x3 l1 l2 l3.
  (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\
  x1 <= x2 /\ x2 <= x3 /\ x3 <= SUC x1 ==>
  ~ALL_DISTINCT (l1 ++ l2 ++ l3)
```

- let's strip the goal

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (``!x1 x2 x3 l1 l2 l3.
  (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\
  ((x1 <= x2) /\ (x2 <= x3) /\ x3 <= SUC x1) ==>
  ~(ALL_DISTINCT (l1 ++ l2 ++ l3))``,
REPEAT STRIP_TAC
```

Tactical Proof - Example II - Slide 2

Current Goal

```
!x1 x2 x3 l1 l2 l3.
  (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\
  x1 <= x2 /\ x2 <= x3 /\ x3 <= SUC x1 ==>
  ~ALL_DISTINCT (l1 ++ l2 ++ l3)
```

- let's strip the goal

Proof Script

```
val LENGTH_APPEND_SAME = prove (
  ``!l. LENGTH (APPEND l l) = 2 * LENGTH l``,
REPEAT STRIP_TAC
```

Actions

- add `REPEAT STRIP_TAC` to proof script
- expand this tactic using hol-mode

Tactical Proof - Example II - Slide 3

Current Goal

```
  0.  MEM x1 l1       4.  x2 <= x3
  1.  MEM x2 l2       5.  x3 <= SUC x1
  2.  MEM x3 l3       6.  ALL_DISTINCT (l1 ++ l2 ++ l3)
  3.  x1 <= x2
------------------------------------
F
```

- oops, we did too much, we would like to keep `ALL_DISTINCT` in goal

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (``...``,
REPEAT GEN_TAC >> STRIP_TAC
```

Actions

- undo `REPEAT STRIP_TAC` `(M-h b)`
- expand more fine-tuned strip tactic

Tactical Proof - Example II - Slide 4

Current Goal

```
  0.  MEM x1 l1        3.  x1 <= x2
  1.  MEM x2 l2        4.  x2 <= x3
  2.  MEM x3 l3        5.  x3 <= SUC x1
------------------------------------
~ALL_DISTINCT (l1 ++ l2 ++ l3)
```

- now let's simplify `ALL_DISTINCT`
- search suitable theorems with `DB.match`
- use them with rewrite tactic

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (``...``,
REPEAT GEN_TAC >> STRIP_TAC >>
REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND]
```

Tactical Proof - Example II - Slide 5

Current Goal

```
  0.  MEM x1 l1        3.  x1 <= x2
  1.  MEM x2 l2        4.  x2 <= x3
  2.  MEM x3 l3        5.  x3 <= SUC x1
------------------------------------
~((ALL_DISTINCT l1 /\ ALL_DISTINCT l2 /\ !e. MEM e l1 ==> ~MEM e l2) /\
  ALL_DISTINCT l3 /\ !e. MEM e l1 \/ MEM e l2 ==> ~MEM e l3)
```

- from assumptions 3, 4 and 5 we know `x2 = x1 \/ x2 = x3`
- let's deduce this fact by `DECIDE_TAC`

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (``...``,
REPEAT GEN_TAC >> STRIP_TAC >>
REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND] >>
`(x2 = x1) \/ (x2 = x3)` by DECIDE_TAC
```

Tactical Proof - Example II - Slide 6

Current Goals — 2 subgoals, one for each disjunct

```
  0.  MEM x1 l1          4.  x2 <= x3
  1.  MEM x2 l2          5.  x3 <= SUC x1
  2.  MEM x3 l3          6a. x2 = x1
  3.  x1 <= x2           6b. x2 = x3
------------------------------------
~((ALL_DISTINCT l1 /\ ALL_DISTINCT l2 /\ !e. MEM e l1 ==> ~MEM e l2) /\
  ALL_DISTINCT l3 /\ !e. MEM e l1 \/ MEM e l2 ==> ~MEM e l3)
```

- both goals are easily solved by first-order reasoning
- let's use METIS_TAC[] for both subgoals

Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (``...``,
REPEAT GEN_TAC >> STRIP_TAC >>
REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND] >>
`(x2 = x1) \/ (x2 = x3)` by DECIDE_TAC >> (
  METIS_TAC[]
));
```

Tactical Proof - Example II - Slide 7

Finished Proof Script

```
val NOT_ALL_DISTINCT_LEMMA = prove (
``!x1 x2 x3 l1 l2 l3.
  (MEM x1 l1 /\ MEM x2 l2 /\ MEM x3 l3) /\
  ((x1 <= x2) /\ (x2 <= x3) /\ x3 <= SUC x1) ==>
  ~(ALL_DISTINCT (l1 ++ l2 ++ l3))``,
REPEAT GEN_TAC >> STRIP_TAC >>
REWRITE_TAC[listTheory.ALL_DISTINCT_APPEND, listTheory.MEM_APPEND] >>
`(x2 = x1) \/ (x2 = x3)` by DECIDE_TAC >> (
  METIS_TAC[]
));
```

- notice that proof structure is explicit
- parentheses and indentation used to mark new subgoals

Part IX

Induction on Numbers and Lists

Mathematical Induction (on Natural Numbers)

- mathematical (a. k. a. natural) induction principle:
  If a property P holds for 0 and $P(n)$ implies $P(n+1)$ for all n, then $P(n)$ holds for all n.
- HOL4 is expressive enough to encode this principle as a theorem.

```
|- !P. P 0 /\ (!n. P n ==> P (SUC n)) ==> !n. P n
```

- Performing mathematical induction in HOL4 means applying this theorem (e. g. via `HO_MATCH_MP_TAC`)

Induction on Lists

- HOL4 lists are defined similarly to in SML:

```
list = NIL | CONS of 'a => list
```

- list induction principle is similar to that of natural numbers:

```
|- !P. P [] /\ (!t. P t ==> !h. P (h::t)) ==> !l. P l
```

- fair warning: these are the **default**, not the only possible induction principles/theorems for numbers and lists

Induction (and Case-Split) Tactics

- the tactic `Induct` (or `Induct_on`) is usually used to start induction proofs
- it looks at the type of the quantifier (or its argument) and applies the default induction theorem for this type
- this is usually what one needs
- similarly, `Cases_on` picks and applies default case-split theorems (e. g. , for `bool`)

List Induction Proof - Example I - Slide 1

- let's prove via induction
  `!l1 l2. REVERSE (l1 ++ l2) = REVERSE l2 ++ REVERSE l1`
- we set up the goal and start an induction proof on `l1`

Proof Script

```
val REVERSE_APPEND = prove (
``!l1 l2. REVERSE (l1 ++ l2) = REVERSE l2 ++ REVERSE l1``,
Induct
```

List Induction Proof - Example I - Slide 2

- the induction tactic produced two cases
- base case:

```
!l2. REVERSE ([] ++ l2) = REVERSE l2 ++ REVERSE []
```

- induction step:

```
!h l2. REVERSE (h::l1 ++ l2) = REVERSE l2 ++ REVERSE (h::l1)
------------------------------------------------------------
!l2. REVERSE (l1 ++ l2) = REVERSE l2 ++ REVERSE l1
```

- both goals can be easily proved by rewriting

Proof Script

```
val REVERSE_APPEND = prove (``
!l1 l2. REVERSE (l1 ++ l2) = REVERSE l2 ++ REVERSE l1``,
Induct >| [
  REWRITE_TAC[REVERSE_DEF, APPEND, APPEND_NIL],
  ASM_REWRITE_TAC[REVERSE_DEF, APPEND, APPEND_ASSOC]
]);
```

List Induction Proof - Example II - Slide 2

- let's prove via induction
  `!l. REVERSE (REVERSE l) = l`
- we set up the goal and start an induction proof on `l`

Proof Script

```
val REVERSE_REVERSE = prove (
``!l. REVERSE (REVERSE l) = l``,
Induct
```

List Induction Proof - Example II - Slide 2

- the induction tactic produced two cases
- base case:

  `REVERSE (REVERSE []) = []`
- induction step:

  ```
  !h. REVERSE (REVERSE (h::l1)) = h::l1
  --------------------------------------------
  REVERSE (REVERSE l) = l
  ```
- again both goals can be easily proved by rewriting

Proof Script

```
val REVERSE_REVERSE = prove (
``!l. REVERSE (REVERSE l) = l``,
Induct >| [
  REWRITE_TAC[REVERSE_DEF],
  ASM_REWRITE_TAC[REVERSE_DEF, REVERSE_APPEND, APPEND]
]);
```