This document is based on material from the “Interactive Theorem Proving Course” by Thomas Tuerk
(`https://www.thomas-tuerk.de`):
`https://github.com/thtuerk/ITP-course`

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Part XV

Advanced Definition Principles

Relations

- a relation is a function from some arguments to `bool`
- the following example types are all types of relations:
  - `: 'a -> 'a -> bool`
  - `: 'a -> 'b -> bool`
  - `: 'a -> 'b -> 'c -> 'd -> bool`
  - `: ('a # 'b # 'c) -> bool`
  - `: bool`
  - `: 'a -> bool`
- relations are closely related to sets
  - `R a b c <=> (a, b, c) IN {(a, b, c) | R a b c}`
  - `(a, b, c) IN S <=> (\a b c. (a, b, c) IN S) a b c`

Relations II

- relations are often defined by a set of **rules**

Definition of Reflexive-Transitive Closure

The reflexive-transitive closure of a relation `R : 'a -> 'a -> bool` can be defined as the least relation `RTC R` that satisfies the following inductive rules:

$$\frac{\texttt{R x y}}{\texttt{RTC R x y}} \qquad \frac{}{\texttt{RTC R x x}} \qquad \frac{\texttt{RTC R x y} \quad \texttt{RTC R y z}}{\texttt{RTC R x z}}$$

- if the rules are monotone, a least and a greatest fixpoint exists (by the Knaster-Tarski theorem)
- least fixpoints give rise to **inductive relations**
- greatest fixpoints give rise to **coinductive relations**

(Co)inductive Relations in HOL4

- `(Co)IndDefLib` provides infrastructure for defining (co)inductive relations
- given a set of rules `Hol_(co)reln` defines (co)inductive relations
- three theorems are returned and stored in current theory:
  - a rules theorem — it states that the defined constant satisfies the rules
  - a cases theorem — this is an equational form of the rules showing that the defined relation is indeed a fixpoint
  - a (co)induction theorem
- additionally, a strong (co)induction theorem is stored in current theory

Example: Reflexive-Transitive Closure

```
> val (RTC_REL_rules, RTC_REL_ind, RTC_REL_cases) = Hol_reln `
    (!x y.   R x y                              ==> RTC_REL R x y) /\
    (!x.                                            RTC_REL R x x) /\
    (!x y z. RTC_REL R x y /\ RTC_REL R y z ==> RTC_REL R x z)`

val RTC_REL_rules = |- !R.
  (!x y. R x y ==> RTC_REL R x y) /\ (!x. RTC_REL R x x) /\
  (!x y z. RTC_REL R x y /\ RTC_REL R y z ==> RTC_REL R x z)

val RTC_REL_cases = |- !R a0 a1.
  RTC_REL R a0 a1 <=>
  (R a0 a1 \/ (a1 = a0) \/ ?y. RTC_REL R a0 y /\ RTC_REL R y a1)
```

Example: Transitive Reflexive Closure II

```
val RTC_REL_ind = |- !R RTC_REL'.
  ((!x y. R x y ==> RTC_REL' x y) /\ (!x. RTC_REL' x x) /\
   (!x y z. RTC_REL' x y /\ RTC_REL' y z ==> RTC_REL' x z)) ==>
  (!a0 a1. RTC_REL R a0 a1 ==> RTC_REL' a0 a1)
```

```
> val RTC_REL_strongind = DB.fetch "-" "RTC_REL_strongind"

val RTC_REL_strongind = |- !R RTC_REL'.
  (!x y. R x y ==> RTC_REL' x y) /\ (!x. RTC_REL' x x) /\
  (!x y z.
      RTC_REL R x y /\ RTC_REL' x y /\ RTC_REL R y z /\
      RTC_REL' y z ==>
      RTC_REL' x z) ==>
  ( !a0 a1. RTC_REL R a0 a1 ==> RTC_REL' a0 a1)
```

Example: EVEN

```
> val (EVEN_REL_rules, EVEN_REL_ind, EVEN_REL_cases) = Hol_reln
  `(EVEN_REL 0) /\ (!n. EVEN_REL n ==> (EVEN_REL (n + 2)))`;

val EVEN_REL_cases =
 |- !a0. EVEN_REL a0 <=> (a0 = 0) \/ ?n. (a0 = n + 2) /\ EVEN_REL n

val EVEN_REL_rules =
 |- EVEN_REL 0 /\ !n. EVEN_REL n ==> EVEN_REL (n + 2)

val EVEN_REL_ind = |- !EVEN_REL'.
   (EVEN_REL' 0 /\ (!n. EVEN_REL' n ==> EVEN_REL' (n + 2))) ==>
   (!a0. EVEN_REL a0 ==> EVEN_REL' a0)
```

- notice that in this example there is exactly one fixpoint
- therefore, for these rules the inductive and coinductive relation coincide

Example: Dummy Relations

```
> val (DF_rules, DF_ind, DF_cases) = Hol_reln
  `(!n. DF (n+1) ==> (DF n))`

> val (DT_rules, DT_coind, DT_cases) = Hol_coreln
  `(!n. DT (n+1) ==> (DT n))`

val DT_coind =
  |- !DT'. (!a0. DT' a0 ==> DT' (a0 + 1)) ==> !a0. DT' a0 ==> DT a0

val DF_ind =
  |- !DF'. (!n. DF' (n + 1) ==> DF' n) ==> !a0. DF a0 ==> DF' a0

val DT_cases = |- !a0. DT a0 <=> DT (a0 + 1):
val DF_cases = |- !a0. DF a0 <=> DF (a0 + 1):
```

- notice that the definitions of `DT` and `DF` look like a non-terminating recursive definition
- `DT` is always true, i. e. `|- !n. DT n`
- `DF` is always false, i. e. `|- !n. ~(DF n)`

Quotient Types

- `quotientLib` allows to define types as quotients of existing types with respect to **partial equivalence relation**
- each equivalence class becomes a value of the new type
- partiality allows ignoring certain values of original type
- `quotientLib` allows to lift definitions and lemmata as well
- details are technical and won't be presented here

Quotient Types Example

- let's assume we have an implementation of finite sets of numbers as binary trees with
  - type `binset`
  - binary tree invariant `WF_BINSET : binset -> bool`
  - constant `empty_binset`
  - add and member functions `add : num -> binset -> binset`, `mem : binset -> num -> bool`
- we can define a partial equivalence relation by

```
binset_equiv b1 b2 := (
  WF_BINSET b1 /\ WF_BINSET b2 /\
  (!n. mem b1 n <=> mem b2 n))
```

- this allows defining a quotient type of sets of numbers
- functions `empty_binset`, `add` and `mem` as well as lemmata about them can be lifted automatically

Quotient Types Summary

- quotient types are sometimes very useful
  - e. g. , rational numbers are defined as a quotient type
  - used extensively by mathematicians
- there is powerful infrastructure for them
- many tasks are automated
- however, the details are technical and won't be discussed here

Pattern Matching / Case Expressions

- pattern matching ubiquitous in functional programming
- pattern matching is a powerful technique
- it helps to write concise, readable definitions
- very handy and frequently used for interactive theorem proving
- however, it is not directly supported by the HOL logic
- representations in HOL4:
  - sets of equations as produced by `Define`
  - decision trees (printed as case-expressions)

TFL / Define

- we have already used top-level pattern matches with the TFL package
- Define is able to handle them
  - all the semantic complexity is taken care of
  - no special syntax or functions remain
  - no special rewrite rules, reasoning tools needed afterwards
- Define produces a set of equations
- this is the **recommended** way of doing pattern matching in HOL4

Example

```
> val ZIP_def = Define `(ZIP (x::xs) (y::ys) = (x,y)::(ZIP xs ys)) /\
                        (ZIP [] [] = [])`
val ZIP_def = |- (!ys y xs x. ZIP (x::xs) (y::ys) = (x,y)::ZIP xs ys) /\
                 (ZIP [] [] = [])
```

Case Expressions

- sometimes one does not want to use this compilation by TFL
  - one wants to use pattern-matches somewhere nested in a term
  - one might not want to introduce a new constant
  - one might want to avoid using TFL for technical reasons
- in such situations, case-expressions can be used
- their syntax is similar to the syntax used by SML

Example

```
> val ZIP_def = Define `ZIP xs ys = case (xs, ys) of
                                  (x::xs, y::ys) => (x,y)::(ZIP xs ys)
                                | ([], []) => []`

val ZIP_def = |- !ys xs. ZIP xs ys =
     case (xs,ys) of
       ([],[]) => []
     | ([],v4::v5) => ARB
     | (x::xs',[]) => ARB
     | (x::xs',y::ys') => (x,y)::ZIP xs' ys'
```

Case Expressions II

- the datatype package defines case-constants for each datatype
- the parser contains a pattern compilation algorithm
- case-expressions are by the parser compiled to decision trees using case-constants
- pretty printer prints these decision trees as case-expressions again

Example

```
val ZIP_def = |- !ys xs. ZIP xs ys =
     pair_CASE (xs,ys)
       (\v v1.
          list_CASE v (list_CASE v1 [] (\v4 v5. ARB))
            (\x xs'. list_CASE v1 ARB (\y ys'. (x,y)::ZIP xs' ys'))):
```

Case Expression Issues

- using case expressions feels very natural to functional programmers
- case-expressions allow concise, well-readable definitions
- however, there are also many drawbacks
- there is large, complicated code in the parser and pretty printer
  - this is outside the kernel
  - parsing a pretty-printed term can result in a non α-equivalent one
  - there are bugs in this code (see e. g. Issue #416 reported 8 May 2017)
- the results are hard to predict
  - heuristics involved in creating decision tree
  - however, it is beneficial that proofs follow this internal, volatile structure

Case Expression Issues II

- technical issues
  - it is tricky to reason about decision trees
  - rewrite rules about case-constants needs to be fetched from `TypeBase`
    - alternative `srw_ss` often does more than wanted
  - partially evaluated decision-trees are not pretty printed nicely any more
- underspecified functions
  - decision trees are exhaustive
  - they list underspecified cases explicitly with value `ARB`
  - this can be lengthy
  - `Define` in contrast hides underspecified cases

Case Expression Example I

Partial Proof Script

```
val _ = prove (``!l1 l2.
  (LENGTH l1 = LENGTH l2) ==>
  ((ZIP l1 l2 = []) <=> ((l1 = []) /\ (l2 = [])))``,

ONCE_REWRITE_TAC [ZIP_def]
```

Current Goal

```
!l1 l2.
  (LENGTH l1 = LENGTH l2) ==>
  (((case (l1,l2) of
       ([],[]) => []
     | ([],v4::v5) => ARB
     | (x::xs',[]) => ARB
     | (x::xs',y::ys') => (x,y)::ZIP xs' ys') =
    []) <=> (l1 = []) /\ (l2 = []))
```

Case Expression Example IIa – partial evaluation

Partial Proof Script

```
val _ = prove (``!l1 l2.
  (LENGTH l1 = LENGTH l2) ==>
  ((ZIP l1 l2 = []) <=> ((l1 = []) /\ (l2 = [])))``,

ONCE_REWRITE_TAC [ZIP_def] >>
REWRITE_TAC[pairTheory.pair_case_def] >> BETA_TAC
```

Current Goal

```
!l1 l2.
  (LENGTH l1 = LENGTH l2) ==>
  (((case l1 of
       [] => (case l2 of [] => [] | v4::v5 => ARB)
     | x::xs' => case l2 of [] => ARB | y::ys' => (x,y)::ZIP xs' ys') =
    []) <=> (l1 = []) /\ (l2 = []))
```

Case Expression Example IIb — following tree structure

Partial Proof Script

```
val _ = prove (``!l1 l2.
  (LENGTH l1 = LENGTH l2) ==>
  ((ZIP l1 l2 = []) <=> ((l1 = []) /\ (l2 = [])))``,

ONCE_REWRITE_TAC [ZIP_def] >>
Cases_on `l1` >| [
  REWRITE_TAC[listTheory.list_case_def]
```

Current Goal

```
!l2.
  (LENGTH [] = LENGTH l2) ==>
  (((case ([],l2) of
       ([],[]) => []
     | ([],v4::v5) => ARB
     | (x::xs',[]) => ARB
     | (x::xs',y::ys') => (x,y)::ZIP xs' ys') =
    []) <=> (l2 = []))
```

Case Expression Summary

- case expressions are natural to functional programmers
- they allow concise, readable definitions
- however, fancy parser and pretty-printer needed
  - trustworthiness issues
  - proving sanity checking lemmas advisable
- reasoning about case expressions can be tricky and lengthy
- proofs about case expressions often hard to maintain
- therefore, use top-level pattern matching via `Define` if possible

Relations and Case Expressions in Practice

- Common uses of relations:
  - well-typing relation for a programming language
  - operational semantics reduction relation of a programming language
  - operational semantics reduction relation of a hardware device
  - proof system rules for a logic
- Common reasoning about relations:
  - if a program is well-typed, it never goes wrong at runtime
  - proof system is sound and and complete
  - whether the well-typing holds or not is decidable

Example: Proof System for Propositional Logic

Propositional Logic Syntax Fragment

$\phi = \phi \wedge \phi \mid p$

Some Propositional Logic Proof Rules

$$\frac{\phi \wedge \psi}{\phi}\ \mathrm{ANDE1} \qquad \frac{\phi \wedge \psi}{\psi}\ \mathrm{ANDE2} \qquad \frac{\phi \quad \psi}{\phi \wedge \psi}\ \mathrm{ANDI}$$

See `https://kth-step.github.io/itppv-course/lectures/propositional.pdf` for more detailed informal definitions that can be directly encoded in HOL4.
Skeleton definitions in HOL4: `https://github.com/kth-step/itppv-course/tree/master/homeworks/hw6-supplementary`

Example: Untyped Lambda Calculus

Lambda Calculus Syntax

$t = x \mid \lambda x.t \mid t\, t'$
$v = \lambda x.t$

Lambda Calculus Semantics

$$\frac{}{(\lambda x.t_1)v_2 \rightarrow \{v_2/x\}t_1}\ \mathrm{AX\_APP} \qquad \frac{t_1 \rightarrow t_1'}{t_1\ t \rightarrow t_1'\ t}\ \mathrm{CTX\_APP\_FUN}$$

$$\frac{t_1 \rightarrow t_1'}{v\ t_1 \rightarrow v\ t_1'}\ \mathrm{CTX\_APP\_ARG}$$

A more detailed informal definition is available at `https://kth-step.github.io/itppv-course/lectures/lambda.pdf`.
The full HOL4 definition is available at `https://github.com/kth-step/itppv-course/tree/master/hol4-examples/untyped-lambda`